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# On locally and globally symmetric Berwald spaces*

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**Abstract**

In this paper, we generalize Cartan's work on Riemannian locally and globally symmetric spaces to locally and globally symmetric Berwald spaces. We prove that a Berwald space is locally symmetric if and only if the flag curvature is invariant under parallel displacements and a locally symmetric Berwald space is locally isometric to a globally symmetric Berwald space.

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## Introduction

The study of Finsler spaces has important applications in physics. In our previous paper [1], we mentioned several aspects of them. Meanwhile, the notion of 'symmetry' is always attracting the attention of physicists and mathematicians. Cartan's work on Riemannian symmetric spaces attains a summit of the study of 'symmetry' in Riemannian geometry. Up to now, it has proved to be the foundation of many new branches in mathematics as well as the applications of geometry to physics. Therefore, it is undoubtedly very important to study the symmetry of Finsler spaces.

Among the Finsler spaces, Berwald spaces are a very important class. A Finsler space is called Berwaldian if the Chern connection defines a linear connection directly on the underlying manifold (cf [2]). Berwald spaces are only a bit more general than Riemannian spaces and locally Minkowskian spaces. They provide examples which are more properly Finslerian, but only slightly so. Since the connection is linear, its tangent spaces are linearly isometric to a common Minkowski space. Therefore Berwald spaces behave very much like Riemannian spaces. Since Riemannian spaces have many applications in certain areas of physics, it is hopeful that Berwald metrics will be useful in the study of some physical problems.

In this paper, we study the symmetry of Berwald spaces. The definition of locally and globally symmetric Berwald spaces was introduced by Szabó [3]. A Berwald space $(M, F)$ is called locally, respectively globally, symmetric if the connection of $(M, F)$ is

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locally, respectively globally (affine), symmetric. Szabó also gave a classification of the manifolds in which there exists an irreducible globally symmetric Berwald metric which is non-Riemannian. His list coincides with Cartan's classification of irreducible globally symmetric Riemannian spaces of rank $\geqslant 2$ (as manifolds). He also pointed out that there exist infinitely many irreducible globally symmetric Berwald metrics on each of the manifolds in his list.

However, there are still some aspects of these manifolds to clarify. First of all, the geometric meaning of these spaces is not clear. In Cartan's definition of Riemannian locally, respectively globally, symmetric spaces, he used the property that geodesic symmetries are isometries, not only affine transformations (cf [5], p 199). Obviously, Cartan's definitions are more closely related to geometry. Secondly, an algebraic description of globally symmetric Berwald spaces is necessary. In his study of Riemannian globally symmetric manifolds, Cartan introduced the notion of orthogonal symmetric Lie algebras to describe such spaces and found that there is a remarkable duality among these spaces. Finally, the relation of locally and globally symmetric Berwald spaces is not clear. Although Szabó proved that each Berwald space (not necessary symmetric) is locally isometric to the product of a locally Minkowski space, a Riemannian space and a globally symmetric Berwald space, it is still not clear whether the product manifold can be chosen to be a globally symmetric Berwald space when the original space is locally symmetric. As a contrary comparison, Cartan proved that each locally symmetric Riemannian space is actually locally isometric to a globally symmetric space.

To solve these problems, we need to study the property of isometries of a Finsler (Berwald) space. In general, we have two ways to define an isometry. The first way is that we call a diffeomorphism $\phi$ of a Finsler manifold $(M, F)$ an isometry if $F(\mathrm{d}\phi(y)) = F(y)$, for any $y \in TM - \{0\}$. On the other hand, we can also define an isometry to be a one-to-one mapping of $M$ onto itself which preserves the distance of each pair of points of $M$. It is well known that these two definitions are equivalent if $F$ is Riemannian (cf [5], chapter 1). In [7] we proved that this is also true for arbitrary Finsler metrics. This result is vital when we study homogeneous Finsler manifolds, as well as the locally, respectively globally, symmetric spaces.

In this paper, we will generalize Cartan's work on Riemannian locally symmetric and globally symmetric spaces to locally symmetric and globally symmetric Berwald spaces. In section 1, we study isometries of Berwald spaces and find a relationship between isometries and affine diffeomorphisms of Berwald spaces. In section 2, we prove that a Berwald space is locally symmetric if and only if the flag curvature is invariant under all parallel displacements. Finally, in section 3, we prove that any locally symmetric Berwald space is locally isometric to a globally symmetric Berwald space. We must point out that our results do not follow directly from Szabó's classification result because he did not classify all the globally symmetric Berwald metrics on each of the manifolds in his list.

## 1. Some results on isometries of Berwald spaces

In this section, we will promote the study of isometries in [7]. This will be useful in the following sections. In this paper, manifolds are always assumed to be connected.

The following results are well known for Riemannian manifolds. However, for a Berwald space, it is far from obvious. The reason is that in a Berwald space the relationship between its metric and connection is not so close as in a Riemannian space.

**Theorem 1.1.** *Let $(M, F)$ be a Berwald space, $\psi$ be an isometry of $(M, F)$ onto itself. Then $\psi$ is an affine transformation with respect to the connection of $F$.*

**Proof.** We first deduce a formula for the connection $D$ of $F$. For any vector fields $T$, $V$, $W$ on $M$, we have (see [2], p 260)

$$Tg_W(V, W) = g_W(D_T V, W) + g_W(V, D_T W). \tag{1.1}$$

Similarly,

$$Vg_W(T, W) = g_W(D_V T, W) + g_W(T, D_V W), \tag{1.2}$$

$$Wg_W(V, W) = g_W(D_W V, W) + g_W(V, D_W W). \tag{1.3}$$

Subtracting (1.2) from the summation of (1.1) and (1.3) we get

$$\begin{aligned} g_W(V, D_{W+T} W) + g_W(W - T, D_V W) &= Tg_W(V, W) - Vg_W(T, W) \\ &\quad + Wg_W(V, W) - g_W([T, V], W) - g_W([W, V], W), \end{aligned}$$

where we have used the symmetry of the connection, i.e., $D_V W - D_W V = [V, W]$. Set $T = W - V$ in the above equation, we obtain

$$2g_W(V, D_W W) = 2Wg_W(V, W) - Vg_W(W, W) - 2g_W([W, V], W). \tag{1.4}$$

Since $\psi$ is an isometry, $\mathrm{d}\psi$ is a linear isometry between the spaces $T_p(M)$ and $T_{\psi(p)}(M)$, $\forall p \in M$. Therefore for any vector fields $X$, $Y$, $Z$ on $M$, we have

$$g_{\mathrm{d}\psi(X)}(\mathrm{d}\psi(Y), \mathrm{d}\psi(Z)) = g_X(Y, Z).$$

By (1.4) we have

$$g_{\mathrm{d}\psi(W)}(\mathrm{d}\psi(V), D_{\mathrm{d}\psi(W)}\,\mathrm{d}\psi(W)) = g_W(V, D_W W).$$

Consequently,

$$g_{\mathrm{d}\psi(W)}(\mathrm{d}\psi(V), D_{\mathrm{d}\psi(W)}\,\mathrm{d}\psi(W)) = g_{\mathrm{d}\psi(W)}(\mathrm{d}\psi(V), \mathrm{d}\psi(D_W W)).$$

Since $V$ is arbitrary and $g_{\mathrm{d}\psi(W)}(\cdot, \cdot)$ is an inner product, we have

$$D_{\mathrm{d}\psi(W)}\,\mathrm{d}\psi(W) = \mathrm{d}\psi(D_W W).$$

Now using the identity

$$D_V W = \tfrac{1}{2}(D_{V+W}(V + W) - D_V V - D_W W - [W, V]),$$

we get

$$D_{\mathrm{d}\psi(V)}\,\mathrm{d}\psi(W) = \mathrm{d}\psi(D_V W).$$

Therefore, $\psi$ is an affine transformation with respect to $D$. □

**Remark.** Although we obtain a formula for $D_W W$, it is generally very difficult to deduce a formula for $D_V W$, because the inner products $g_V(\cdot, \cdot)$ and $g_W(\cdot, \cdot)$ are different.

**Theorem 1.2.** *Let $(M_i, F_i)$, $i = 1, 2$ be two Berwald spaces, $\psi$ be an affine diffeomorphism from $(M_1, F_1)$ onto $(M_2, F_2)$ with respect to the connections of $F_1$ and $F_2$. If there exists $p \in M$ such that* $\mathrm{d}\psi$ *is a linear isometry from $T_p(M_1)$ onto $T_{\psi(p)}(M_2)$, then $\psi$ is an isometry.*

**Proof.** Let $q \in M$. We only need to prove that $F_1(y) = F_2(\mathrm{d}\psi(y))$, $\forall y \in T_q(M)$. Join $q$ to $p$ by a curve $\gamma$. Let $\tau$ denote the parallel transformation from $q$ to $p$ along $\gamma$. Then for any $u, v \in T_q(M)$, we have

$$g_y(u, v) = g_{\tau(y)}(\tau(u), \tau(v)) = g_{\mathrm{d}\psi(\tau(y))}(\mathrm{d}\psi(\tau(u)), \mathrm{d}\psi(\tau(v))).$$

Now $\tau(y)$, $\tau(u)$, $\tau(v)$ is the result of the parallel displacement (along $\gamma$) of $y$, $u$, $v$, respectively. Since $\psi$, being an affine diffeomorphism, transforms vectors that are parallel along $\gamma$ into vectors that are parallel along $\psi \cdot \gamma$. Therefore $\mathrm{d}\psi(\tau(y)), \mathrm{d}\psi(\tau(u)), \mathrm{d}\psi(\tau(v))$ must be the result of the parallel displacement (along $\psi \cdot \gamma$) of $\mathrm{d}\psi(y), \mathrm{d}\psi(u), \mathrm{d}\psi(v)$, respectively. Thus

$$g_{\mathrm{d}\psi(\tau(y))}(\mathrm{d}\psi(\tau(u)), \mathrm{d}\psi(\tau(v))) = g_{\mathrm{d}\psi(y)}(\mathrm{d}\psi(u), \mathrm{d}\psi(v)).$$

Therefore

$$g_y(u, v) = g_{\mathrm{d}\psi(y)}(\mathrm{d}\psi(u), \mathrm{d}\psi(v)).$$

This completes the proof because $F_1(y) = \sqrt{g_y(y, y)}$ and

$$F_2(\mathrm{d}\psi(y)) = \sqrt{g_{\mathrm{d}\psi(y)}(\mathrm{d}\psi(y), \mathrm{d}\psi(y))}.$$ □

## 2. Geometric meaning of locally symmetric and globally symmetric Berwald spaces

In this section, we will clarify the geometric meaning of locally and globally symmetric Berwald spaces. We first recall a definition. Let $M$ be a Finsler space, $p \in M$. Then there exists a neighbourhood $N_0$ of the origin of the tangent space $T_p(M)$ such that the exponential mapping $\exp_p$ is a ($C^1$, and $C^\infty$ if $M$ is Berwald) diffeomorphism of $N_0$ onto a neighbourhood $N_p$ of $p$ in $M$ (cf [2]). We can also assume that $N_0 = -N_0$. Now we define a mapping of $N_p$ onto itself by

$$s_p : \exp(y) \to \exp(-y), \qquad y \in N_0.$$

Then $s_p$ is called the geodesic symmetry with respect to $p$. $M$ is called locally geodesic symmetric if for any $p \in M$, there exists $N_p$ such that $s_p$ is an isometry of $N_p$.

**Proposition 2.1.** *A locally geodesic symmetric Berwald space $(M, F)$ must be locally symmetric. If $F$ is absolutely homogeneous, then the converse is also true.*

**Proof.** The first conclusion is a direct consequence of theorem 1.1. If $F$ is absolutely homogeneous and $(M, F)$ is locally symmetric, then for any $p \in M$, we can find a neighbourhood $N_p$ of $p$ such that $s_p$ is an affine transformation. Note that $(\mathrm{d}s_p)_p = -I$ and $F$ is absolutely homogeneous, so $(\mathrm{d}s_p)_p$ is a linear isomorphism of $T_p(M)$. By theorem 1.2, $s_p$ is an isometry. Thus $(M, F)$ is locally geodesic symmetric. □

**Remark.** A locally symmetric Berwald space may not be locally geodesic symmetric. In fact, any Minkowski space is locally symmetric (since the curvature tensor vanishes). But if the Minkowski norm is not absolutely homogeneous, then it is not locally geodesic symmetric. Another example is given by $S^2 \times S^1$. In [2], the authors defined a Randers metric $F$ on $S^2 \times S^1$ (p 306). It is easy to see that this metric is locally symmetric (in fact, the connection of $F$ is globally affine symmetric) but not locally geodesic symmetric. In [7], we give another explanation of this metric as an invariant Berwald metric on $SO(3) \times S^1/(SO(2) \times \{e\})$. From this explanation, we can also see that this Berwald metric is globally symmetric. But it is not geodesic symmetric (Randers metrics are not absolutely homogeneous unless they are Riemannian).

The following result is a generalization of Cartan's work on Riemannian manifolds. However, the proof is more complicated. Let us first explain the notion of parallel displacements of flags. Let $(M, F)$ be a Berwald space and $(P, y)$ be a flag in a tangent space $T_p(M)$, $p \in M$. Let $q \in M$ and $c(t)$ be a piecewise smooth curve in $M$ connecting $p$ and $q$ and $\tau$ be the parallel displacement along $c(t)$. Then $\tau(P)$ is a plane in $T_q(M)$ (since $\tau$ is a linear isomorphism) and $\tau(y) \neq 0, \tau(y) \in \tau(P)$. Therefore $(\tau(P), \tau(y))$ is a flag in $T_q(M)$. We say that the flag curvature is invariant under the parallel displacement $\tau$ if $K(P, y) = K(\tau(P), \tau(y))$, for any flag $(P, y)$ in $T_p(M)$.

**Theorem 2.2.** *Let $(M, F)$ be a Berwald space. Then M is locally symmetric if and only if the flag curvature is invariant under all parallel displacements.*

**Proof.** Let $(M, F)$ be a locally symmetric Berwald space. Then its connection is locally affine symmetric. So the curvature tensor $R$ is invariant under all parallel displacements (cf [5], p 198). On the other hand, let $p, q \in M$, $\gamma$ be a curve joining $p$ to $q$, and $\tau$ be the parallel displacement along $\gamma$. Then $\forall y\ (\neq 0), u, v \in T_p(M)$, we have $g_{\tau(y)}(\tau(u), \tau(v)) = g_y(u, v)$. Therefore, by the definition of flag curvature, we have

$$K(P, y) = K(\tau(P), \tau(y)),$$

where $P$ is any plane in $T_p(M)$ containing $y$. That is, the flag curvature is invariant under all parallel displacements. Conversely, suppose that the flag curvature is invariant under all parallel displacements. Let $p, q, \gamma, \tau, y, u, v$ be as above and suppose that $u$ is linearly independent of $y$. Let $l = \frac{y}{F}$. Consider the quantity (cf [2], p 69)

$$K(l, u, v) = \frac{g_l(R(l, u)l, v)}{g_l(u, v) - g_l(l, u)g_l(l, v)}. \tag{2.1}$$

Then $K(P, y) = K(l, u, u)$. Since $K(P, y)$ is invariant under parallel displacements, we have

$$K(l, u, u) = K(\tau(l), \tau(u), \tau(u)).$$

By the polarization identity (cf [2], p 70)

$$K(l, u, v) = \tfrac{1}{4}(K(l, u + v, u + v) - K(l, u - v, u - v)),$$

we have $K(l, u, v) = K(\tau(l), \tau(u), \tau(v))$. Since in (2.1) the denominator is invariant under parallel displacements, we have

$$g_l(R_p(l, u)l, v) = g_{\tau(l)}(R_q(\tau(l), \tau(u))\tau(l), \tau(v)).$$

Therefore

$$g_{\tau(l)}(\tau(R_p(l, u)l), \tau(v)) = g_{\tau(l)}(R_q(\tau(l), \tau(u))\tau(l), \tau(v)).$$

This means

$$\tau(R_p(l, u)l) = R_q(\tau(l), \tau(u))\tau(l). \tag{2.2}$$

It is obvious that the equality still holds if $u$ is linearly dependent on $y$. Now we use a result of Szabó [3] which asserts that for the Berwald space $(M, F)$ there exists a Riemannian metric $g$ on $M$ with the same connection as $F$. Consider the quadrilinear form $B$ defined by

$$B(u, v, z, t) = g(R_q(\tau(u), \tau(v))\tau(z), \tau(t)) - g(\tau(R_p(u, v)z), \tau(t)),$$

where $u, v, z, t \in T_p(M)$. Then we have

(a) $B(u, v, z, t) = -B(v, u, z, t)$;
(b) $B(u, v, z, t) = -B(u, v, t, z)$;
(c) $B(u, v, z, t) + B(v, z, u, t) + B(z, u, v, t) = 0$;
(d) $B(u, v, u, v) = 0$.

In fact (a) is the well-known property of the curvature tensor of a Riemannian manifold. (b) and (c) follow from the fact

$$g(\tau(R_p(u, v)z), \tau(t)) = g(R_p(u, v)z, t),$$

since $\tau$ is also the parallel displacement with respect to $g$. And (d) follows from (2.2). By a well-known result in Riemannian geometry, we have $B \equiv 0$. Thus

$$\tau(R_p(u, v)z) = R_q(\tau(u), \tau(v))\tau(z),$$

i.e., $\tau R_p = R_q$. Therefore $D_U R = 0$ for each vector field $U$. Thus $D$ is locally affine symmetric. □

Similarly, we call a Berwald space $(M, F)$ a globally geodesic symmetric Berwald space if it is locally geodesic symmetric and for each $p \in M$, the geodesic symmetry $s_p$ can be extended to an isometry of $M$.

**Theorem 2.3.** *Let $(M, F)$ be a Berwald space. Then*

*(a) $(M, F)$ is globally symmetric if and only if each point in $M$ is the isolated fixed point of an involutive affine transformation of $(M, F)$.*
*(b) $(M, F)$ is globally geodesic symmetric if and only if each point in $M$ is the isolated fixed point of an involutive isometry of $(M, F)$.*
*(c) A globally geodesic symmetric Berwald space must be globally symmetric. If the metric is absolutely homogeneous, then the converse is also true.*

**Proof.** (a) is standard (cf [4], pp 222–9). To prove (b), it is sufficient to prove that an involutive isometry with isolated fixed point must be a geodesic symmetry. This can be proved similarly as in [5] (p 205). (c) is the consequence of (a) and (b). □

## 3. Locally and globally symmetric Berwald spaces

Now we can prove

**Theorem 3.1.** *Let $(M, F)$ be a locally symmetric Berwald space. Then for any $p \in M$ there exists a globally symmetric Berwald space $(\tilde{M}, \tilde{F})$, a neighbourhood $N_p$ of $p$ in $M$ and an isometry $\phi$ of $N_p$ onto an open neighbourhood of $\phi(p)$ in $\tilde{M}$. Furthermore, if $(M, F)$ is locally geodesic symmetric, then $(\tilde{M}, \tilde{F})$ can be chosen to be globally geodesic symmetric.*

**Proof.** Let $D$ denote the connection of $F$. Then Szabó proved that there exists a Riemannian metric $g$ on $M$ such that $D$ is the Levi-Civita connection of $g$ [3]. It is well known that $(M, g)$ is a Riemannian locally symmetric space [5]. Thus there exists a Riemannian globally symmetric space $(\tilde{M}, \tilde{g})$, a neighbourhood $N_p$ of $p$ in $M$, an isometry (with respect to $g$ and $\tilde{g}$) $\varphi$ of $N_p$ onto a neighbourhood of $\varphi(p)$ in $\tilde{M}$ ([5], theorem 5.1). Let $\tilde{D}$ be the Levi-Civita connection of $\tilde{g}$. Let $H$ and $\tilde{H}$ denote the holonomy group of $D$ (at $p$) and $\tilde{D}$ (at $\tilde{p} = \varphi(p)$), respectively. Then $\mathrm{d}\varphi_p$ induces an isomorphism between the holonomy algebra $\mathfrak{h}$ of $H$ and $\tilde{\mathfrak{h}}$ of $\tilde{H}$. Hence there exists a neighbourhood $U_e$ of the unit element $e$ of $H$ and a neighbourhood $\tilde{U}_{\tilde{e}}$ of the unit element $\tilde{e}$ of $\tilde{H}$ such that $\mathrm{d}\varphi_p$ induces a local isomorphism between $U_e$ and $\tilde{U}_{\tilde{e}}$. Without losing generality, we can assume that $\tilde{M}$ is simply connected ([5], corollary 5.7). Then $\tilde{H}$ is connected and is generated by the elements of $\tilde{U}_{\tilde{e}}$.

Now we define a Berwald metric on $\tilde{M}$. Identifying $T_{\tilde{p}}(\tilde{M})$ with $T_p(M)$ through $\mathrm{d}\varphi_p$, we get a Minkowski norm $\tilde{F}$ on $T_{\tilde{p}}(\tilde{M})$ by

$$\tilde{F}(\tilde{x}) = F((\mathrm{d}\varphi_p)^{-1}(\tilde{x})), \qquad \tilde{x} \in T_{\tilde{p}}(\tilde{M}).$$

Since $F$ is invariant under $H$, $\tilde{F}$ is invariant under $\tilde{U}_{\tilde{e}}$. Therefore, using the fact that $\tilde{H}$ is generated by the elements of $\tilde{U}_{\tilde{e}}$, we see that $\tilde{F}$ is invariant under $\tilde{H}$. Now for any $\tilde{q} \in \tilde{M}$, we join $\tilde{q}$ to $\tilde{o}$ by a curve $\tilde{\gamma}$. Let $\tilde{\tau}_{\tilde{\gamma}}$ be the parallel displacement of $\tilde{D}$ along $\tilde{\gamma}$. We then define a Finsler metric (still denoted by $\tilde{F}$) on $\tilde{M}$ by

$$\tilde{F}(\tilde{u}) = \tilde{F}(\tilde{\tau}_{\tilde{\gamma}}(\tilde{u})), \qquad \tilde{u} \in T_{\tilde{q}}(\tilde{M}).$$

It is easily seen that $\tilde{F}$ is well defined and $(\tilde{M}, \tilde{F})$ is a Berwald space with connection $\tilde{D}$, so it is a globally symmetric Berwald space.

It remains to prove that $\varphi$ is an isometry. But this follows easily from the fact that $\varphi$ is an affine diffeomorphism and $\mathrm{d}\varphi_p$ is a linear isometry (see theorem 1.2).

The last conclusion is easy to verify. □

**Corollary 3.2.** *Let $(M, F)$ be a locally symmetric Berwald space, and $(\tilde{M}, \pi)$ be the universal covering manifold of $M$. Then $\tilde{M}$ with the metric $\pi^*(F)$ is a globally symmetric Berwald space.*

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